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The structure of the anomalies of gauge theories in the causal approach

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Abstract

We consider the gauge invariance of the standard Yang–Mills model in the framework of the causal approach of Epstein–Glaser and Scharf and determine the generic form of the anomalies. The method used is based on the Epstein–Glaser approach to renormalization theory. In the case of quantum electrodynamics we obtain quite easily the absence of anomalies in all orders.

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1. Introduction

The causal approach to renormalization theory of by Epstein and Glaser [31, 32] had produced important simplification of the renormalization theory at the purely conceptual level as well as to the computational aspects. This approach works for quantum electrodynamics (QED) [22, 37, 53] where it brings important simplifications of the renormalizability proof. For Yang–Mills theories [3, 5, 17, 18, 24, 25, 27–29, 34–36, 41–44, 46, 55] one can determine severe constraints on the interaction Lagrangian (or in the language of the renormalization theory—on the first-order chronological product) from the condition of gauge invariance. Gravitation can be also analysed in this framework [33, 39, 40, 62], etc. Finally, the analysis of scale invariance can be performed [38, 51]. One should stress the fact that the Epstein–Glaser analysis uses exclusively the Bogoliubov axioms of renormalization theory [12] imposed on the scattering matrix: this is an operator acting in the Hilbert space of the model, usually generated from the vacuum by the quantum fields. If one considers the S-matrix as a perturbative expansion in the coupling constant of the theory, one can translate these axioms on the chronological products. Epstein–Glaser approach is a inductive procedure to construct the chronological products in higher orders starting from the first order of perturbation theory. For gauge theories one can construct a non-trivial interaction only if one considers a larger Hilbert space generated by the fields associated with the particles of the model and the ghost fields. The condition of gauge invariance becomes in this framework the condition of factorization of the S-matrix to the physical Hilbert space in the adiabatic limit. To avoid infra-red problems one works with a formulation of this factorization condition which corresponds to a formal adiabatic limit and it is perfectly rigorously defined [25] and exploited in detail in [17, 24–30]. The obstructions to the implementation of the condition of gauge invariance are called *anomalies*. The most famous is the Adler–Bell–Bardeen–Jackiw anomaly [1, 6, 7, 10] (see [50] for a review). We mention the fact that in the Epstein–Glaser approach to renormalization theory one starts with a fixed Fock space (in which the scattering matrix lives); this means that the masses of the particles are fixed from the very beginning. This means that one can describe the phenomena of radiative mass generation only if one works with interacting fields.

The classical analysis of the renormalizability of Yang–Mills theories of Becchi *et al* [11] is based on a different combinatorial idea. Namely, one considers a perturbative expansion in Planck constant $\hbar$ which is equivalent, in Feynman graphs terminology, to a loop expansion. (The rigorous connection between these two perturbation schemes has been recently under investigation [23].) One can formulate the condition of gauge invariance in terms of the generating functional for the one-particle irreducible Feynman amplitudes; the S-matrix is then recovered using the reduction formulæ [52]. Presumably, both formulations lead to the same S-matrix, up to finite renormalization, although this point is not firmly established in the literature. The most difficult part is to prove that if there are no anomalies in lower orders of perturbation theory, then the anomalies are absent in higher orders. The main tool of the proof is the consideration of the scale invariance properties of a quantum theory expressed in the form of Callan–Symanzik equations [13–15]. A mathematical analysis was developed in [60] and [61], using the quantum action principle [48] (for a review see [52]). One should stress the fact that in this approach one works with interaction fields which can be defined as formal series in the coupling constant. The main observation used in these references is the existence of anomalous dimensional behaviour of the (interacting) fields with respect to dilations. Based on this analysis in [9] (see also [11] and [52]) it is showed that the ABBJ anomaly can appear only in the order $n = 3$ of the perturbation theory. A analysis of the standard model based on this approach can be found in [47].

In [38] we have investigated scale anomaly from the point of view of Epstein–Glaser causal approach using the perturbation scheme of Bogoliubov based on an expansion in the coupling constant. We have found out the surprising result that scale invariance does not restrict the presence of the anomalies in higher orders of perturbation theory. So, from the point of view of Bogoliubov axioms, the elimination of anomalies in higher orders of perturbation theory is still an open question.

The purpose of this paper is to investigate the generic form of the anomalies compatible with the restrictions following from covariance properties and formal gauge invariance. Our analysis is in the same spirit as the traditional analysis based on the Wess–Zumino consistency conditions satisfied by the generating functional of the 1PI Feynman amplitudes [63]. The classification of the anomalies is considered an important point in the usual BRST approach of the renormalization program so we think that the same problem deserves a thorough analysis also in the Epstein–Glaser–Scharf formulation. We must mention here that for the case of QED and Yang–Mills theories with massless bosons (with Dirac matter) the absence of the anomalies (in the Epstein–Glaser approach) can be proved by different methods (see [26, 53] and [24, 25, 27, 28] respectively). For the case of QED a related proof appeared in [37]. Our strategy will be based exclusively on the Epstein–Glaser construction of the chronological products for the free fields. The role of Feynman graph combinatorics is completely eliminated in this analysis. We will use in fact a reformulation of the Epstein–Glaser formalism [22, 59] which gives a prescription for the construction of the chronological products of the type $T(W_1(x_1), \ldots, W_n(x_n))$ for any Wick polynomials $W_1(x_1), \ldots, W_n(x_n)$. The main point is to formulate a proper induction hypothesis for the expression $d_Q T(W_1(x_1), \ldots, W_n(x_n))$ where

d_Q is the free BRST operator [21]. In fact, it is necessary to make such a conjecture only for some special cases of Wick polynomials: this idea was first implemented in [19] for a pure Yang–Mills model, without Dirac fermions and without symmetry breaking. If $T(x)$ is the interaction Lagrangian (i.e. the first-order chronological product) one can prove the validity of some 'descent' equations of the type

$$\begin{aligned} &d_Q T(x) = \mathrm{i}\partial_\mu T^\mu(x) && d_Q T^\mu(x) = \mathrm{i}\partial_\nu T^{\mu\nu}(x), \ldots \\ &d_Q T^{\mu_1,\ldots,\mu_{p-1}}(x) = \mathrm{i}\partial_{\mu_p} T^{\mu_1,\ldots,\mu_p}(x) && d_Q T^{\mu_1,\ldots,\mu_p}(x) = 0. \end{aligned} \tag{1.1}$$

In the QED the procedure stops after the first step ($p = 1$) and in the Yang–Mills case after a two steps ($p = 2$). In general, one can consider the case when the descent stops after a finite number of steps. In this case one has to give a proper conjecture for $d_Q T(W_1(x_1), \ldots, W_n(x_n))$ only for $W_1(x_1), \ldots, W_n(x_n)$ of the type $T(x), T^\mu(x), T^{\mu\nu}(x), \ldots$. This was done in [19] (see also [30] the footnote of p 4328); in our notations this conjecture has the form (3.14) and (4.12).

In two previous papers [35, 36] we have showed that the elimination of the anomalies in the second and third order of the perturbation theory gives important restrictions on the parameters of the generalized Yang–Mills model. In particular, one has severe restrictions on the gauge group. The goal is to prove that no anomalies can appear in higher orders of the perturbation theory, probably for $n \geqslant 6$ as it is implied by the some arguments presented in [50]. We prove here that this goal cannot be attained by purely algebraic methods based on consistency conditions of Wess–Zumino type. However, if some additional assumptions are made one can prove the absence of anomalies; this is the case for a pure Yang–Mills theory with zero-mass bosons with $SU(N)$-invariance and no axial couplings of the Dirac matter fields. This model was analysed in detail in [28] and [17] where the absence of anomalies was proved using a charge conjugation invariance argument. We can show that the same argument works in our approach also. So we may conclude that the main interest of our paper resides in the investigation of the limits of the Wess–Zumino analysis of the anomalies in the framework of the causal approach. We derive the explicit form of the Wess–Zumino consistency relations in Epstein–Glaser–Scharf formulation of gauge models and we derive the most general form of the anomalies from pure algebraic considerations (Lorentz covariance, symmetry and power counting arguments). We work in the most general case of a model with massive and massless bosons and with matter fields. We prove the absence of anomalies in the case of QED and quantum chromodynamics (QCD) with $SU(N)$-invariance and no axial couplings; in this way we rederive the results from [28] and [17] in a different way. For the general case, like in the traditional approaches to renormalization theory, the problem remains open and the best one can do is to hope for some non-renormalization theorems for higher orders of the perturbation theory. This will be analysed in a forthcoming paper.

The structure of this paper is the following one. In the next section we make a brief review of essential points concerning Epstein–Glaser resolution scheme of Bogoliubov axioms and the standard model in the framework of the causal approach (for more details see [37] and [35]). We emphasize that the main problem is to establish the factorization of the S-matrix to the physical Hilbert space; in the formal adiabatic limit, this is the famous condition of gauge invariance. Translated in terms of Feynman amplitudes this condition amounts, essentially, to the so-called Ward–Takahashi identities, or—in the language of the Zürich group—the Cg identities. In the next section we give the inductive hypothesis for QED and prove that there are no anomalies. Next, we do the same thing for Yang–Mills theories. We use the conjecture that the gauge invariance condition has the generic form given by the relation (4.12) (according to [19]) and determine the generic form of the anomalies. When we particularize to the case treated in [27,28] and [17], i.e. when only massless bosons are present we can see that we have essentially reobtained the results from these papers in a purely algebraic way.

2. Perturbation theory in the causal approach

We give here the essential ingredients of perturbation theory.

2.1. Bogoliubov axioms

We present the point of view of of Stora and Fredenhagen [8, 22, 59]; the main objects are the chronological products. An equivalent point of view uses retarded products [58]. By a *perturbation theory* in the sense of Bogoliubov we mean an ensemble of operator-valued distributions $T(W_1(x_1), \ldots, W_n(x_n))$ $n = 1, 2, \ldots$ acting in some Fock space and called *chronological products* (where $W_1(x_1), \ldots, W_n(x_n)$ are arbitrary Wick monomials) verifying the following set of axioms:

- Skew symmetry in all arguments $W_1(x_1), \ldots, W_n(x_n)$:

  $$T(\ldots, W_i(x_i), W_{i+1}(x_{i+1}), \ldots,) = (-1)^{f_i f_{i+1}} T(\ldots, W_{i+1}(x_{i+1}), W_i(x_i), \ldots) \tag{2.1}$$

  where f_i is the number of Fermi fields appearing in the Wick monomial W_i.
- Poincaré invariance: for all $(a, A) \in$ in $SL(2, \mathbb{C})$ we have

  $$U_{a,A} T(W_1(x_1), \ldots, W_n(x_n)) U_{a,A}^{-1} \\ = T(A \cdot W_1(\delta(A) \cdot x_1 + a), \ldots, A \cdot W_n(\delta(A) \cdot x_n + a)). \tag{2.2}$$

  Sometimes it is possible to supplement this axiom by corresponding invariance properties with respect to inversions (spatial and temporal) and charge conjugation. For the standard model only the PCT invariance is available. Also some other global symmetry with respect to some internal symmetry group might be imposed.
- Causality: if $x_i \geqslant x_j, \forall \mathrm{i} \leqslant k,\ j \geqslant k+1$ then we have

  $$T(W_1(x_1), \ldots, W_n(x_n)) = T(W_1(x_1), \ldots, W_k(x_k)) T(W_{k+1}(x_{k+1}), \ldots, W_n(x_n)). \tag{2.3}$$

- Unitarity: we define the *anti-chronological products* according to

  $$(-1)^n \bar{T}(W_1(x_1), \ldots, W_n(x_n)) \equiv \sum_{r=1}^{n} (-1)^r \sum_{I_1, \ldots, I_r \in \mathrm{Part}\,(\{1, \ldots, n\})} \epsilon\ T_{I_1}(X_1) \cdots T_{I_r}(X_r) \tag{2.4}$$

  where we have used the notation

  $$T_{\{i_1, \ldots, i_k\}}(x_{i_1}, \ldots, x_{i_k}) \equiv T(W_{i_1}(x_{i_1}), \ldots, W_{i_k}(x_{i_k})) \tag{2.5}$$

  and the sign ϵ counts the permutations of the Fermi factors. Then the unitarity axiom is

  $$\bar{T}(W_1(x_1), \ldots, W_n(x_n)) = T(W_1^*(x_1), \ldots, W_n^*(x_n)). \tag{2.6}$$

- The 'initial condition'

  $$T(W(x)) = W(x). \tag{2.7}$$

Remark 2.1. From (2.3) one can derive easily that if we have $x_i \sim x_j, \forall \mathrm{i} \leqslant k,\ j \geqslant k+1$ then

$$[T(W_1(x_1), \ldots, W_k(x_k)), T(W_{k+1}(x_{k+1}), \ldots, W_n(x_n))]_{\mp} = 0. \tag{2.8}$$

One extends the definition of $T(W_1(x_1), \ldots, W_n(x_n))$ for $W_1(x_1), \ldots, W_n(x_n)$ Wick polynomials by linearity.

It can be proved that this system of axioms can be supplemented with the normalization condition of the type

$$T(W_1(x_1), \ldots, W_n(x_n)) = \sum \epsilon \langle \Omega, T(W_1'(x_1), \ldots, W_n'(x_n))\Omega\rangle : W_1''(x_1), \ldots, W_n''(x_n)) \tag{2.9}$$

where W_i' and W_i'' are Wick submonomials of W_i such that $W_i = \, :W_i' W_i'':$ the sign ϵ takes care of the permutation of the Fermi fields and Ω is the vacuum state.

We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated with arbitrary Wick monomials $W_1, \ldots, W_n$; explicitly

$$\omega(\langle \Omega, T(W_1(x_1), \ldots, W_n(x_n))\Omega\rangle) \leqslant \sum_{l=1}^{n} \omega(W_l) - 4(n-1) \tag{2.10}$$

where by $\omega(d)$ we mean the order of singularity of the (numerical) distribution d and by $\omega(W)$ we mean the canonical dimension of the Wick monomial W.

2.2. Massive Yang–Mills fields

In [34–36] we have justified the following scheme for the standard model (SM): we consider the auxiliary Hilbert space $\mathcal{H}^{gh,r}_{\mathrm{YM}}$ generated from the vacuum Ω by applying the free fields $A_{a\mu},\ u_a,\ \tilde{u}_a,\ \Phi_a \quad a = 1, \ldots, r$ where the first one has vector transformation properties with respect to the Poincaré group and the others are scalars. In other words, every vector field has three scalar partners. Also $u_a,\ \tilde{u}_a \quad a = 1, \ldots, r$ are fermion and $A_\mu,\ \Phi_a \quad a = 1, \ldots, r$ are boson fields.

We have two distinct possibilities for distinct indices a:

(I) Fields of type I correspond to an index a such that the vector field A_a^μ has non-zero mass m_a. In this case we suppose that all the other scalar partners fields $u_a,\ \tilde{u}_a,\ \Phi_a$ have the same mass m_a.

(II) Fields of type II correspond to an index a such that the vector field A_a^μ has zero mass. In this case we suppose that the scalar partners fields $u_a,\ \tilde{u}_a$ also have the zero mass but the scalar field Φ_a can have a non-zero mass: $m_a^H \geqslant 0$. It is convenient to use the compact notation

$$m_a^* \equiv \begin{cases} m_a & \text{for } m_a \neq 0 \\ m_a^H & \text{for } m_a = 0. \end{cases} \tag{2.11}$$

Then the following equations of motion describe the preceding construction:

$$\begin{aligned} &(\square + m_a^2)u_a(x) = 0 \qquad && (\square + m_a^2)\tilde{u}_a(x) = 0 \\ &(\square + (m_a^*)^2)\Phi_a(x) = 0 \qquad && a = 1, \ldots, r. \end{aligned} \tag{2.12}$$

We also postulate the following canonical (anti)commutation relations:

$$\begin{aligned} &[A_{a\mu}(x), A_{b\nu}(y)] = -\delta_{ab} g_{\mu\nu} D_{m_a}(x-y) \times \mathbf{1} \\ &\{u_a(x), \tilde{u}_b(y)\} = \delta_{ab} D_{m_a}(x-y) \times \mathbf{1} \qquad [\Phi_a(x), \Phi_b(y)] = \delta_{ab} D_{m_a^*}(x-y) \times \mathbf{1} \end{aligned} \tag{2.13}$$

all other (anti)commutators are null.

In this Hilbert space we suppose given a sesquilinear form $\langle \cdot, \cdot \rangle$ such that

$$\begin{aligned} &A_{a\mu}(x)^\dagger = A_{a\mu}(x) \qquad && u_a(x)^\dagger = u_a(x) \\ &\tilde{u}_a(x)^\dagger = -\tilde{u}_a(x) \qquad && \Phi_a(x)^\dagger = \Phi_a(x). \end{aligned} \tag{2.14}$$

The ghost degree is ± 1 for the fields u_a (resp. $\tilde{u}_a$), $a = 1, \ldots, r$ and 0 for the other fields.

One can define the BRST *gauge charge* Q by

$$\begin{aligned} &\{Q, u_a\} = 0 && \{Q, \tilde{u}_a\} = -\mathrm{i}(\partial_\mu A_a^\mu + m_a \Phi_a) \\ &[Q, A_a^\mu] = \mathrm{i}\partial^\mu u_a && [Q, \Phi_a] = \mathrm{i} m_a u_a \qquad \forall a = 1, \ldots, r \end{aligned} \tag{2.15}$$

and

$$Q\Omega = 0. \tag{2.16}$$

Then one can justify that the *physical* Hilbert space of the Yang–Mills system is a factor space

$$\mathcal{H}^r_{\mathrm{YM}} \equiv \mathcal{H} \equiv \mathrm{Ker}(Q)/\mathrm{Ran}(Q). \tag{2.17}$$

The sesquilinear form $\langle\cdot,\cdot\rangle$ induces a *bona fide* scalar product on the Hilbert factor space.

The factorization process leads to the following *physical* particle content of this model:

- For $m_a > 0$ the fields $A_a^\mu,\ u_a,\ \tilde{u}_a,\ \Phi_a$ describe a particle of mass $m_a > 0$ and spin 1; these are the so-called *heavy bosons* [35].
- For $m_a = 0$ the fields $A_a^\mu,\ u_a,\ \tilde{u}_a$ describe a particle of mass 0 and helicity 1; the typical example is the *photon* [34].
- For $m_a = 0$ the fields Φ_a describe a scalar field of mass m_a^H; these are the so-called *Higgs fields*.

This framework is sufficient for the study of the SM of the electro-weak interactions. To include also QCD one must consider that there is a third case:

(III) Fields of type III correspond to an index a such that the vector field A_a^μ has zero mass, the scalar partners $u_a,\ \tilde{u}_a$ also have zero mass but the scalar field Φ_a is absent.

In [55] and [30] the model is constructed somewhat differently: one eliminates the fields of type II and includes a number of supplementary scalar bosonic fields φ_i of masses $m_i \geqslant 0$. In this framework one can consider for instance the very interesting Higgs–Kibble model in which there are no zero-mass particle, so the adiabatic limit probably exists.

We can preserve the general framework with only two types of indices if we consider that in case II there are in fact three subcases (i.e three types of indices a for which $m_a = 0$):

(IIa) In this case $A_{a\mu},\ u_a,\ \tilde{u}_a,\ \Phi_a \not\equiv 0$;

(IIb) In this case $\Phi_a \equiv 0$;

(IIc) In this case $A_{a\mu},\ u_a,\ \tilde{u}_a \equiv 0$.

One must modify appropriately the canonical (anti)commutation relations (2.13) to avoid contradiction for some values of the indices. One has some freedom of notation: for instance, one can eliminate case (IIa) if one includes the first three fields fields in case (IIb) and the last one in case (IIc). The relations (2.15) are not affected in this way.

Let us consider the set of Wick monomials $\mathcal{W}$ constructed from the free fields $A_a^\mu,\ u_a,\ \tilde{u}_a$ and Φ_a for all indices $a = 1, \ldots, r$; we define the BRST operator $d_Q : \mathcal{W} \to \mathcal{W}$ as the (graded) commutator with the gauge charge operator Q. Then one can prove easily that

$$d_Q^2 = 0. \tag{2.18}$$

The class of observables on the factor space is defined as follows: an operator $O : \mathcal{H}^{gh,r}_{\mathrm{YM}} \to \mathcal{H}^{gh,r}_{\mathrm{YM}}$ induces a well defined operator $[O]$ on the factor space $\overline{\mathrm{Ker}(Q)/\mathrm{Im}\,(Q)} \simeq \mathcal{F}_m$ if and only if it verifies

$$d_Q O|_{\mathrm{Ker}(Q)} = 0. \tag{2.19}$$

Because of the relation (2.18) not all operators verifying the condition (2.19) are interesting. In fact, the operators of the type $d_Q O$ are inducing a null operator on the factor space; explicitly, we have

$$[d_Q O] = 0. \tag{2.20}$$

We will construct a perturbation theory verifying Bogoliubov axioms using this set of free fields and imposing the usual axioms of causality, unitarity and relativistic invariance on the chronological products $T(x_1, \ldots, x_n)$. Moreover, we want that the result factorizes to the physical Hilbert space in the formal adiabatic limit. This amounts to [3, 29]:

$$d_Q T(x_1, \ldots, x_n) = \mathrm{i} \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} T_l^\mu(x_1, \ldots, x_n) \tag{2.21}$$

for some auxiliary chronological products $T_l^\mu(x_1, \ldots, x_n), l = 1, \ldots, n$ which must be determined recurringly, together with the standard chronological products.

If one adds matter fields we proceed as before. In particular, we suppose that the BRST operator acts trivially on the matter fields. It seems that the matter field must be described by a set of Dirac fields of masses M_A, $A = 1, \ldots, N$ denoted by $\psi_A(x)$. These fields are characterized by the following relations [36]; here $A, B = 1, \ldots, N$.

The equation of motion

$$(\mathrm{i}\gamma \cdot \partial - M_A)\psi_A(x) = 0. \tag{2.22}$$

Canonical (anti)commutation relations

$$\{\psi_A(x), \overline{\psi_B}(y)\} = \delta_{AB} S_{M_A}(x - y) \tag{2.23}$$

and all other (anti)commutators are null.

By a *trivial Lagrangian* we mean a Wick expression of the type

$$L(x) = d_Q N(x) + \mathrm{i} \frac{\partial}{\partial x^\mu} L^\mu(x) \tag{2.24}$$

with $L(x)$ and $L^\mu(x)$ some Wick polynomials. The first term in the previous formula gives zero by factorization to the physical Hilbert space (according to a previous discussion) and the second one gives also zero in the adiabatic limit; this justifies the elimination of such expression from the first-order chronological product $T(x)$.

One can prove [35,36] that the condition (2.21) for $n = 1, 2, 3$ determines quite drastically the interaction Lagrangian (up to a trivial Lagrangian):

$$\begin{aligned} T(x) \equiv & -f_{abc}[\tfrac{1}{2} : A_{a\mu}(x) A_{b\nu}(x) F_c^{\mu\nu}(x) : + : A_a^\mu(x) u_b(x) \partial_\mu \tilde{u}_c(x) :] \\ & + f'_{abc}[: \Phi_a(x) \partial_\mu \Phi_b(x) A_c^\mu(x) : - m_b : \Phi_a(x) A_{b\mu}(x) A_c^\mu(x) : \\ & + m_b : \Phi_a(x) \tilde{u}_b(x) u_c(x) :] \\ & + f''_{abc} : \Phi_a(x) \Phi_b(x) \Phi_c(x) : + j_a^\mu(x) A_{a\mu}(x) + j_a(x) \Phi_a(x). \end{aligned} \tag{2.25}$$

Here

$$F_a^{\mu\nu}(x) \equiv \partial^\mu A_a^\nu(x) - \partial^\nu A_a^\mu(x) \tag{2.26}$$

is the Yang–Mills field tensor and the so-called *currents* are

$$j_a^\mu(x) =: \overline{\psi_A}(x)(t_a)_{AB} \gamma^\mu \psi_B(x) : + : \overline{\psi_A}(x)(t'_a)_{AB} \gamma^\mu \gamma_5 \psi_B(x) : \tag{2.27}$$

$$j_a(x) =: \overline{\psi_A}(x)(s_a)_{AB} \psi_B(x) : + : \overline{\psi_A}(x)(s'_a)_{AB} \gamma_5 \psi_B(x) : \tag{2.28}$$

and a number of restrictions must be imposed on the various constants (see [34–36] where the condition of gauge invariance is analysed up to order 3). In particular the constants f_{abc} are completely antisymmetric and verify Jacobi identity. It follows [63] that they generate a compact reductive Lie algebra.

Moreover, we can take $T^\mu(x)$ to be

$$\begin{aligned} T^\mu(x) = & f_{abc}[: u_a(x) A_{b\nu}(x) F_c^{\nu\mu}(x) : - \tfrac{1}{2} : u_a(x) u_b(x) \partial^\mu(x) \tilde{u}_c(x) :] \\ & + f'_{abc}[m_a : A_a^\mu(x) \Phi_b(x) u_c(x) : + : \Phi_a(x) \partial^\mu \Phi_b(x) u_c(x) :] + u_a(x) j_a^\mu(x). \end{aligned} \tag{2.29}$$

The expressions $T(x)$ and $T^\mu(x)$ are $SL(2,\mathbb{C})$-covariant, are causally commuting and are Hermitean. Moreover we have the following ghost content:

$$gh(T(x)) = 0 \qquad gh(T^\mu(x)) = 1. \tag{2.30}$$

Remark 2.2. The presence of indices of type (IIb) and (IIc) is taken into account by requiring that the constants from $T(x)$ are null if one of the indices a, b, c takes such values.

3. The renormalizability of quantum electrodynamics

3.1. The general setting

The case of QED is a particular case of the scheme described in the preceding section. We have only one field of type (IIb), i.e. the triplet $A_\mu,\ u,\ \tilde{u}$ of null mass; they describe a system of null-mass bosons of helicity 1 (i.e. *photons*). We also have only one Dirac field ψ describing the electron. We suppose that in the Hilbert space $\mathcal{H}^{gh}$ generated by these fields from the vacuum Ω we also have a sesqui-linear form $\langle\cdot,\cdot\rangle$ and we denote the conjugate of the operator O with respect to this form by $O^\dagger$. We characterize this form by requiring

$$A_\mu(x)^\dagger = A_\mu(x) \qquad u(x)^\dagger = u(x) \qquad \tilde{u}(x)^\dagger = -\tilde{u}(x). \tag{3.1}$$

The unitary operator realizing the charge conjugation is defined by

$$\begin{aligned} &U_C A^\mu(x) U_C^{-1} = -A^\mu(x) \qquad U_C u(x) U_C^{-1} = -u(x) \qquad U_C \tilde{u}(x) U_C^{-1} = -\tilde{u}(x) \\ &U_C \psi(x) U_C^{-1} = \gamma_0\gamma_2 \bar{\psi}(x)^T \qquad U_C\Omega = \Omega. \end{aligned} \tag{3.2}$$

Now, we define in $\mathcal{H}^{gh}$ the *supercharge* according to

$$Q\Omega = 0 \tag{3.3}$$

and

$$\{Q, u(x)\} = 0 \qquad \{Q, \tilde{u}(x)\} = -\mathrm{i}\partial^\mu A_\mu(x) \qquad [Q, A_\mu(x)] = \mathrm{i}\partial_\mu u(x). \tag{3.4}$$

The expression of the BRST-operator d_Q follows as a particular case of the corresponding formulæ of the Yang–Mills case. From these properties one can derive

$$Q^2 = 0 \tag{3.5}$$

so we also have

$$\mathrm{Im}\,(Q) \subset \mathrm{Ker}(Q). \tag{3.6}$$

By definition, the interaction Lagrangian is

$$T(x) \equiv e : \bar{\psi}(x)\gamma_\mu\psi(x) : A^\mu(x) \tag{3.7}$$

(here e is a real constant: the electron charge) and one can verify easily that we have the covariance properties with respect to $SL(2,\mathbb{C})$. The most important property is (2.21) for $n = 1$:

$$d_Q T(x) = \mathrm{i}\frac{\partial}{\partial x^\mu} T^\mu(x) \tag{3.8}$$

with

$$T^\mu(x) \equiv e : \bar{\psi}(x)\gamma^\mu\psi(x) : u(x). \tag{3.9}$$

One can easily check that we have charge-conjugation invariance in the sense

$$U_C T(x) U_C^{-1} = T(x) \qquad U_C T^\mu(x) U_C^{-1} = T^\mu(x). \tag{3.10}$$

We note that we also have

$$d_Q T^\mu(x) = 0. \tag{3.11}$$

3.2. The main result

It is convenient to write the formulæ (3.8) and (3.11) in a compact way as follows. One denotes by $A^k(x)$, $k = 1, \ldots, 5$ the expressions $T(x)$, $T^\mu(x)$; that is, the index k can take the values L, μ according to the identification: $A^L(x) \equiv T(x)$, $A^\mu(x) \equiv T^\mu(x)$. Then we can write (3.8) and (3.11) in the compact form

$$d_Q A^k(x) = \mathrm{i} \sum_{m=1}^{5} c_m^{k;\mu} \frac{\partial}{\partial x^\mu} A^m(x) \qquad k = 1, \ldots, 5 \tag{3.12}$$

for some constants $c_m^{k;\mu}$; the explicit expressions can be obtained from the corresponding gauge conditions. Only the expressions

$$c_\nu^{L;\mu} \equiv \delta_\nu^\mu \tag{3.13}$$

are non-zero. Then we can prove the following result:

Theorem 3.1. *One can chose the chronological products such that, beside the fulfilment of the Bogoliubov axioms, the following identities are verified:*

$$d_Q T(A^{k_1}(x_1), \ldots, A^{k_p}(x_p)) = \mathrm{i} \sum_{l=1}^{p} (-1)^{s_l} \sum_m c_m^{k_l;\mu} \frac{\partial}{\partial x_l^\mu} T(A^{k_1}(x_1), \ldots, A^m(x_l), \ldots, A^{k_p}(x_p)) \tag{3.14}$$

for all $p \in \mathbb{N}$ *and all* $k_1, \ldots, k_p = 1, \ldots, 5$. *Here we have denoted*

$$s_0 \equiv 0 \qquad s_l \equiv \sum_{j=1}^{l-1} gh(A_j) \qquad \forall l = 1, \ldots, p. \tag{3.15}$$

Proof. (i) We use induction. Suppose we have constructed the chronological products such that that all conditions are satisfied up to order $p = n - 1$. One can construct the chronological products in order n such that all Bogoliubov axioms are satisfied, except the condition of gauge invariance. This can be done directly from the Epstein–Glaser methods [37] or using the extension method [22]. We include in the induction hypothesis the following form for the chronological products:

$$T(T^{\mu_1}(x_1), \ldots, T^{\mu_p}(x_p), T(x_{p+1}), \ldots, T(x_n)) = :u(x_1)\ldots u(x_p): \times \sum_{I,J,K} :\prod_{i\in I} \bar\psi(x_i) t_{I,J,K}^{\mu_1,\ldots,\mu_p;\rho_K}(X) \prod_{j\in J} \psi(x_j) :: \prod_{k\in K} A_{\rho_k}(x_k): \tag{3.16}$$

where: (a) the sums runs over all distinct triplets $I, J, K \subset \{1, \ldots, n\}$ verifying $|I| = |J|$ and $1, \ldots, p \notin K$; (b) by ρ_K we mean the set $\{\rho_k\}_{k\in K}$; (c) the expression $t_{I,J,K}^{\mu_1,\ldots,\mu_p;\rho_K}$ are numerical distributions; more precisely, they take values in the matrix space $M_{\mathbb{C}}(4,4)^{\otimes|I|}$ and one can suppose natural symmetry properties.

In particular we have

$$gh(T(A^{k_1}(x_1), \ldots, A^{k_p}(x_p))) = \sum_{l=1}^{n} gh(A^{k_l}). \tag{3.17}$$

Moreover, the symmetry axiom implies relations of the type

$$T(A_1(x_1), A_2(x_2), \ldots) = (-1)^{gh(A_1)gh(A_2)} T(A_2(x_2), A_1(x_1), \ldots). \tag{3.18}$$

From the induction procedure (or the extension method) one can easily prove the possible obstructions to the gauge invariance condition (3.14) in order n have a particular structure. We have

$$d_Q T(A^{k_1}(x_1), \ldots, A^{k_n}(x_n))$$
$$= \mathrm{i} \sum_{l=1}^{n} (-1)^{s_l} \sum_m c_m^{k_l;\mu} \frac{\partial}{\partial x_l^\mu} T(A^{k_1}(x_1), \ldots, A^m(x_l), \ldots, A^{k_n}(x_n))$$
$$+ P^{k_1,\ldots,k_n}(x_1, \ldots, x_n) \tag{3.19}$$

where $P^{\cdots}(X) \equiv P^{\cdots}(x_1, \ldots, x_n)$ are quasi-local operators called *anomalies*. They have the following structure:

$$P(X) = \sum_L [p_L(\partial)\delta(X)]\, W_L(X) \tag{3.20}$$

where W_L are Wick monomials and p_L are polynomials in the derivatives of the type

$$p_L(X) = \sum_{|\alpha| \leqslant \deg(p_L)} c_{L,\alpha} \partial^\alpha \tag{3.21}$$

with the maximal degree restricted by

$$\deg(p_L) + \omega_L \leqslant 5. \tag{3.22}$$

Moreover, we can easily obtain

$$gh(P^{k_1,\ldots,k_n}) = \sum_{l=1}^{n} gh(A^{k_l}) + 1. \tag{3.23}$$

From (3.16) one can easily see that the anomalies depend only on the fields

$$A_\mu, u, \partial_\nu u, \psi, \partial_\nu \psi, \bar{\psi}, \partial_\nu \bar{\psi}. \tag{3.24}$$

Finally, the anomalies can be chosen $SL(2, \mathbb{C})$-covariant and charge conjugation invariant

$$U_C P^{k_1,\ldots,k_n}(X) U_C^{-1} = P^{k_1,\ldots,k_n}(X). \tag{3.25}$$

(ii) We have a lot of restrictions on the anomalies. The most sever one comes from (3.22) and (3.23): we obtain that for

$$\sum_{l=1}^{n} gh(A^{k_l}) \geqslant 5 \tag{3.26}$$

there are no anomalies. From this restriction it follows that we have the following set of relations with possible anomalies:

$$d_Q T(T(x_1), \ldots, T(x_n))$$
$$= \mathrm{i} \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} T(T(x_1), \ldots, T^\mu(x_l), \ldots, T(x_n)) + P_1(x_1, \ldots, x_n) \tag{3.27}$$

$$d_Q T(T^\mu(x_1), T(x_2), \ldots, T(x_n))$$
$$= -\mathrm{i} \sum_{l=2}^{n} \frac{\partial}{\partial x_l^\nu} T(T^\mu(x_1), T(x_2), \ldots, T^\nu(x_l), \ldots, T(x_n)) + P_2^\mu(x_1, \ldots, x_n) \tag{3.28}$$

$$d_Q T(T^\mu(x_1), T^\nu(x_2), T(x_3), \ldots, T(x_n))$$
$$= \mathrm{i} \sum_{l=3}^{n} \frac{\partial}{\partial x_l^\rho} T(T^\mu(x_1), T^\nu(x_2), T(x_3), \ldots, T^\rho(x_l), \ldots, T(x_n))$$
$$+ P_3^{\mu\nu}(x_1, \ldots, x_n) \tag{3.29}$$

$$d_Q T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \ldots, T(x_n))$$
$$= -\mathrm{i} \sum_{l=4}^{n} \frac{\partial}{\partial x_l^\sigma} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \ldots, T^\sigma(x_l), \ldots, T(x_n))$$
$$+ P_4^{\mu\nu\rho}(x_1, \ldots, x_n) \tag{3.30}$$
$$d_Q T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), \ldots, T(x_n))$$
$$= \mathrm{i} \sum_{l=5}^{n} \frac{\partial}{\partial x_l^\lambda} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), \ldots, T^\lambda(x_l), \ldots, T(x_n))$$
$$+ P_5^{\mu\nu\rho\lambda}(x_1, \ldots, x_n) \tag{3.31}$$

where we use, as before, the convention $\sum_\emptyset \equiv 0$. We can assume that

$$\begin{aligned} P_3^{\mu\nu}(X) &= 0 \qquad |X| = 1 \\ P_4^{\mu\nu\rho}(X) &= 0 \qquad |X| = 2 \\ P_5^{\mu\nu\rho}(X) &= 0 \qquad |X| = 3 \end{aligned} \tag{3.32}$$

without losing generality. The anomalies verify the restrictions (3.22) and (3.23) and they depend only on the fields (3.24).

From (3.18), we get the following symmetry properties:

$P_1(x_1, \ldots, x_n)$ is symmetric in $x_1, \ldots, x_n$ (3.33)

$P_2^\mu(x_1, \ldots, x_n)$ is symmetric in $x_2, \ldots, x_n$ (3.34)

$P_3^{\mu\nu}(x_1, \ldots, x_n)$ is symmetric in $x_3, \ldots, x_n$ (3.35)

$P_4^{\mu\nu\rho}(x_1, \ldots, x_n)$ is symmetric in $x_4, \ldots, x_n$ (3.36)

$P_5^{\mu\nu\rho\sigma}(x_1, \ldots, x_n)$ is symmetric in $x_5, \ldots, x_n$ (3.37)

$P_3^{\mu\nu}(x_1, \ldots, x_n)$ is antisymmetric in $(x_1, \mu), (x_2, \nu)$ (3.38)

$P_4^{\mu\nu\rho}(x_1, \ldots, x_n)$ is antisymmetric in $(x_1, \mu), (x_2, \nu), (x_3, \rho)$ (3.39)

$P_5^{\mu\nu\rho\sigma}(x_1, \ldots, x_n)$ is antisymmetric in $(x_1, \mu), (x_2, \nu), (x_3, \rho), (x_4, \sigma)$. (3.40)

(iii) If we apply the operator d_Q to the anomalous relations (3.27)–(3.31) we easily obtain some consistency relations quite analogous to the well known Wess–Zumino consistency relations:

$$d_Q P_1(x_1, \ldots, x_n) = -\mathrm{i} \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} P_2^\mu(x_l, x_1, \ldots, \hat{x}_l, \ldots, x_n) \tag{3.41}$$

$$d_Q P_2^\mu(x_1, \ldots, x_n) = \mathrm{i} \sum_{l=2}^{n} \frac{\partial}{\partial x_l^\nu} P_3^{\mu\nu}(x_1, x_l, x_2, \ldots, \hat{x}_l, \ldots, x_n) \tag{3.42}$$

$$d_Q P_3^{\mu\nu}(x_1, \ldots, x_n) = -\mathrm{i} \sum_{l=3}^{n} \frac{\partial}{\partial x_l^\rho} P_4^{\mu\nu\rho}(x_1, x_2, x_l, x_3, \ldots, \hat{x}_l, \ldots, x_n) \tag{3.43}$$

$$d_Q P_4^{\mu\nu\rho}(x_1, \ldots, x_n) = \mathrm{i} \sum_{l=4}^{n} \frac{\partial}{\partial x_l^\sigma} P_5^{\mu\nu\rho\sigma}(x_1, x_2, x_3, x_l, x_4, \ldots, \hat{x}_l, \ldots, x_n) \tag{3.44}$$

$$d_Q P_5^{\mu\nu\rho\sigma}(x_1, \ldots, x_n) = 0. \tag{3.45}$$

We will use repeatedly the identity

$$\sum_{l=1}^{n} \frac{\partial}{\partial x_l^\rho} \delta(X) = 0 \tag{3.46}$$

where

$$\delta(X) \equiv \delta(x_1 - x_n) \cdots \delta(x_{n-1} - x_n). \tag{3.47}$$

If we take into account (3.22) and (3.23), the generic form of the anomalies is

$$\begin{aligned} P_l^{\cdots}(X) &= \delta(X)\tilde{W}_l^{\cdots}(x_1) + \sum_{p=1}^{n}\left[\frac{\partial}{\partial x_p^\mu}\delta(X)\right]\tilde{W}_{l;p}^{\ldots;\mu}(X) + \sum_{p,q=1}^{n}\left[\frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu}\delta(X)\right]\tilde{W}_{l;pq}^{\ldots;\mu\nu}(X) \\ &\quad + \sum_{p,q,r=1}^{n}\left[\frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho}\delta(X)\right]\tilde{W}_{l;pqr}^{\ldots;\mu\nu\rho}(X) \\ &\quad + \sum_{p,q,r,s=1}^{n}\left[\frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho \partial x_s^\sigma}\delta(X)\right]\tilde{W}_{l;pqrs}^{\ldots;\mu\nu\rho\sigma}(X) \end{aligned} \tag{3.48}$$

where $\tilde{W}_{\cdots}^{\cdots}$ are some Wick polynomials with convenient symmetry properties. If we use (3.46) we can eliminate all derivatives with respect to one variable, say x_1 if we redefine conveniently the expressions $\tilde{W}_{\cdots}^{\cdots}$:

$$\begin{aligned} P_l^{\cdots}(X) &= \delta(X)W_l^{\cdots}(x_1) + \sum_{p=2}^{n}\frac{\partial}{\partial x_p^\mu}\delta(X)W_{l;p}^{\ldots;\mu}(x_1) + \sum_{p,q=2}^{n}\frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu}\delta(X)W_{l;pq}^{\ldots;\mu\nu}(x_1) \\ &\quad + \sum_{p,q,r=2}^{n}\frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho}\delta(X)W_{l;pqr}^{\ldots;\mu\nu\rho}(x_1) \\ &\quad + \sum_{p,q,r,s=2}^{n}\frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho \partial x_s^\sigma}\delta(X)W_{l;pqrs}^{\ldots;\mu\nu\rho\sigma}(x_1). \end{aligned} \tag{3.49}$$

If $f \in \mathcal{S}(\mathbb{R}^{4n})$ is arbitrary we have

$$\begin{aligned} \langle P_l^{\cdots}, f(X)\rangle &= \int \mathrm{d}x\, f(x,\ldots,x)W_l^{\cdots}(x) - \sum_{p=2}^{n}\int \mathrm{d}x\, (\partial_\mu^p f)(x,\ldots,x)W_{l;p}^{\ldots;\mu}(x) \\ &\quad + \sum_{p,q=2}^{n}\int \mathrm{d}x\, (\partial_\mu^p \partial_\nu^q f)(x,\ldots,x)W_{l;pq}^{\ldots;\mu\nu}(x) + \cdots. \end{aligned} \tag{3.50}$$

But the expressions $f(x,\ldots,x)$, $(\partial_\mu^p f)(x,\ldots,x)$, $(\partial_\mu^p \partial_\nu^q f)(x,\ldots,x), \ldots \quad p,q,\ldots \geqslant 2$ can be chosen arbitrarily, so we have

$$P_l^{\cdots}(X) \quad\Longleftrightarrow\quad W_l^{\cdots} = 0 \qquad W_{l;p}^{\ldots;\mu}(X) = 0 \qquad W_{l;pq}^{\ldots;\mu\nu}(X) = 0. \tag{3.51}$$

As a consequence, every symmetry property

$$\langle P_l^{\cdots}(X), f^g(X)\rangle = \langle P_l^{\cdots}(X), f(X)\rangle \tag{3.52}$$

for g an arbitrary symmetry, will be equivalent to corresponding symmetry properties for the Wick polynomials:

$$g \cdot W = W. \tag{3.53}$$

(iv) Let us consider $l = 3, 4, 5$; because in this case $gh(P_l^{\cdots}) \geqslant 3$ in every Wick polynomial $W_{\cdots}^{\cdots}$ from (3.49) we have at least two factors u (the third can be ∂u) so we get quite easily that

$$P_l^{\cdots}(X) = 0 \qquad l = 3, 4, 5. \tag{3.54}$$

In the cases $l = 1, 2$ we can still simplify the expressions of the anomalies by finite renormalizations. We will indicate the corresponding analysis below. We note that from the consistency conditions (3.41)–(3.45) only the first two are non-trivial.

(v) Because $gh(P_2) = 2$ the generic expression of P_2 is

$$P_2^\mu(X) = \delta(X) W_2^\mu(x_1) + \sum_{p=2}^n \frac{\partial}{\partial x_p^\nu} \delta(X) W_{2;p}^{\mu;\nu}(x_1) + \sum_{p,q=2}^n \frac{\partial^2}{\partial x_p^\nu \partial x_q^\rho} \delta(X) W_{2;pq}^{\mu;\nu\rho}(x_1) \tag{3.55}$$

(because the contribution with three partial derivatives corresponds to the Wick monomial $W_{l;pqr}^{\mu;\nu\rho\sigma} \sim \ : uu : \ = 0$). If we use the symmetry property (3.34) we get

$$\begin{aligned} W_{2;p}^{\mu;\nu} &= W_{2;2}^{\mu;\nu} \equiv W_2^{\mu;\nu} \qquad && \forall p = 2, \ldots, n \\ W_{2;pq}^{\mu;\nu\rho} &= W_{2;22}^{\mu;\nu\rho} \equiv W_2^{\mu;\nu\rho} \qquad && \forall p, q = 2, \ldots, n \end{aligned} \tag{3.56}$$

so we can write the preceding expression more simply:

$$P_2^\mu(X) = \delta(X) W_2^\mu(x_1) + \sum_{p=2}^n \frac{\partial}{\partial x_p^\nu} \delta(X) W_2^{\mu;\nu}(x_1) + \sum_{p,q=2}^n \frac{\partial^2}{\partial x_p^\nu \partial x_q^\rho} \delta(X) W_2^{\mu;\nu\rho}(x_1). \tag{3.57}$$

If we use (3.46) then after some relabelling we obtain

$$P_2^\mu(X) = \delta(X) W_2^\mu(x_1) + \frac{\partial}{\partial x_1^\nu} [\delta(X) W_2^{\mu;\nu}(x_1)] + \frac{\partial^2}{\partial x_1^\nu \partial x_1^\rho} [\delta(X) W_2^{\mu;\nu\rho}(x_1)] \tag{3.58}$$

and we can assume that

$$W_2^{\mu;\nu\rho} = (\nu \leftrightarrow \rho). \tag{3.59}$$

The consistency relation (3.42) becomes equivalent to

$$d_Q W_2^\mu = 0 \qquad d_Q W_2^{\mu;\nu} = 0 \qquad d_Q W_2^{\mu;\nu\rho} = 0. \tag{3.60}$$

The generic form of $W_2^{\mu;\nu\rho}$ is

$$W_2^{\mu;\nu\rho} = c_1^{\mu\nu\rho\sigma} : u \partial_\sigma u : \tag{3.61}$$

with $c_1^{\mu\nu\rho\sigma}$ a Lorentz tensor. If we define

$$U_2^{\mu;\nu\rho} = c_1^{\mu\nu\rho\sigma} : u A_\sigma : \tag{3.62}$$

then we have

$$d_Q U_2^{\mu;\nu\rho} = -\mathrm{i} W_2^{\mu;\nu\rho}. \tag{3.63}$$

It follows that if we perform the finite renormalization:

$$T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) \to T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) + \mathrm{i} \frac{\partial^2}{\partial x_1^\nu \partial x_1^\rho} \left[\delta(X) U_2^{\mu\nu\rho}(x_1) \right] \tag{3.64}$$

we do not change the symmetry properties and the field structure. As a result we make

$$W_2^{\mu;\nu\rho} = 0. \tag{3.65}$$

In the same way, we have the generic expression

$$W_2^{\mu;\nu} = c_2^{\mu\nu\rho\sigma} : u \partial_\rho u A_\sigma : \tag{3.66}$$

with $c_2^{\mu\nu\rho\sigma}$ a Lorentz invariant tensor. If we use charge conjugation invariance we obtain that this expression must vanish. Another way to prove this is to obtain from (3.41) antisymmetry in the first two indices and from the second equation (3.60) symmetry in the last two indices. All these restrictions lead to $c_2^{\mu\nu\rho\sigma} = 0$.

Finally we have the generic form

$$W_2^\mu = d_1 : u \partial^\mu u : + d_2 : u \partial^\mu u A_\rho A^\rho : + d_3 : u \partial_\rho u A_\rho A^\mu : \tag{3.67}$$

for some constants d_i. The first equation (3.60) gives $d_3 = 2d_1$. If we define

$$U_2^\mu = d_1 : u A^\mu : + d_2 : u A^\mu u A_\rho A^\rho : \tag{3.68}$$

we get

$$d_Q U_2^\mu = \mathrm{i} W_2^\mu. \tag{3.69}$$

Now we perform the finite renormalization

$$T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) \to T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) + \mathrm{i}\delta(X) U_2^\mu(x_1) \tag{3.70}$$

we do not affect the properties of the chronological products, we do not spoil the previous finite renormalization and we make

$$P_2^\mu = 0. \tag{3.71}$$

(vi) Because $gh(P_1) = 1$ we have the generic expression

$$\begin{aligned} P_1(X) = {} & \delta(X) W_1(x_1) + \sum_{p=2}^n \frac{\partial}{\partial x_p^\mu} \delta(X) W_{1;p}^\mu(x_1) + \sum_{p,q=2}^n \frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu} \delta(X) W_{1;pq}^{\mu\nu}(x_1) \\ & + \sum_{p,q,r=2}^n \frac{\partial^3}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho} \delta(X) W_{1;pqr}^{\mu\nu\rho}(x_1) \\ & + \sum_{p,q,r,s=2}^n \frac{\partial^4}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho \partial x_s^\sigma} \delta(X) W_{1;pqrs}^{\mu\nu\rho\sigma}(x_1). \end{aligned} \tag{3.72}$$

The symmetry requirement (3.33) in $x_2, \ldots, x_n$ leads as above at a simpler form:

$$\begin{aligned} P_1(X) = {} & \delta(X) W_1(x_1) + \frac{\partial}{\partial x_1^\mu} \left[\delta(X) W_1^\mu(x_1)\right] + \frac{\partial^2}{\partial x_1^\mu \partial x_1^\nu} \left[\delta(X) W_1^{\mu\nu}(x_1)\right] \\ & + \frac{\partial^3}{\partial x_1^\mu \partial x_1^\nu \partial x_1^\rho} \left[\delta(X) W_1^{\mu\nu\rho}(x_1)\right] + \frac{\partial^4}{\partial x_1^\mu \partial x_1^\nu \partial x_1^\rho \partial x_1^\sigma} \left[\delta(X) W_1^{\mu\nu\rho\sigma}(x_1)\right] \end{aligned} \tag{3.73}$$

and the Wick polynomials have convenient symmetry properties. We have the generic form

$$W_1^{\mu\nu\rho\sigma} = c_3^{\mu\nu\rho\sigma} u \tag{3.74}$$

with $c_3^{\mu\nu\rho\sigma}$ a Lorentz covariant tensor. If we perform the finite renormalization

$$\begin{aligned} T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) \to {} & T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) \\ & + \mathrm{i} \frac{\partial^3}{\partial x_1^\nu \partial x_1^\rho \partial x_1^\sigma} \left[\delta(X) W_1^{\mu\nu\rho\sigma}(x_1)\right] \end{aligned} \tag{3.75}$$

we do not affect the symmetry properties and the field dependence (3.24), we do not spoil the previous two finite renormalizations, but as a result we eliminate the last term in the expression of P_1. We now impose the symmetry property (3.33) in x_1, x_2 and obtain that for $n \geqslant 3$

$$P_1(X) = \delta(X) W_1(x_1) \tag{3.76}$$

and for $n = 2$

$$P_1(X) = \delta(X) W_1(x_1) + \frac{\partial}{\partial x_1^\mu} \left[\delta(X) W_1^\mu(x_1)\right] + \frac{\partial^2}{\partial x_1^\mu \partial x_1^\nu} \left[\delta(X) W_1^{\mu\nu}(x_1)\right] \tag{3.77}$$

where

$$W_1^\mu = -\partial_\nu W_1^{\mu\nu} \tag{3.78}$$

and the Wick monomial $W_1^{\mu\nu}$ can be chosen symmetric in the Lorentz indices.

Next, we use the consistency condition (3.41) and get

$$d_Q W_1 = 0 \qquad d_Q W_1^{\mu\nu} = 0. \tag{3.79}$$

The generic form of $W_1^{\mu\nu}$ is

$$W_1^{\mu\nu} = c_3^{\mu\nu\rho\sigma} : uA_\rho A_\sigma : + c_4^{\mu\nu\rho\sigma} : \partial_\rho u A_\sigma : \tag{3.80}$$

with $c_i^{\mu\nu\rho\sigma}$ being Lorentz invariant tensors symmetric in the first two indices. From the last equation of (3.79) we obtain that the first contribution is zero and the second tensor is antisymmetric in the last two indices. The Lorentz covariance makes this tensor also zero so $W_1^{\mu\nu} = 0$.

The generic form of W_1 is

$$\begin{aligned} W_1 = c_1 u + c_2 : uA_\mu A^\mu : &+ c_3 : \partial_\mu u A^\mu : + c_4 : u\bar{\psi}\psi : + c_5 : u\bar{\psi}\gamma_5\psi : + c_6 : uA_\mu\bar{\psi}\gamma^\mu\psi : \\ &+ c_7 : uA_\mu\bar{\psi}\gamma^\mu\gamma_5\psi : + c_8 : \partial_\mu u A^\mu A_\rho A^\rho : + c_9 : uA^\mu A^\mu A_\rho A^\rho : \end{aligned} \tag{3.81}$$

Now it is time to use charge conjugation invariance of the anomalies (3.25) for P_1; we easily get $c_i = 0, i = 1, 2, 4, 5, 7, 9$. If we impose the first condition (3.79) we get $c_6 = 0$. It follows that we are left with

$$W_1 = c_3 : \partial_\mu u A^\mu : + c_9 : \partial_\mu u A^\mu A_\rho A^\rho : \tag{3.82}$$

If we define

$$U_1 \equiv \tfrac{1}{2} c_3 : A_\mu A^\mu : + \tfrac{1}{4} c_9 : A_\mu A^\mu A_\rho A^\rho : \tag{3.83}$$

we have

$$d_Q U_1 = \mathrm{i} W_1. \tag{3.84}$$

Finally, we preform the finite renormalization:

$$T(T(x_1), \ldots, T(x_n)) \to T(T(x_1), \ldots, T(x_n)) + \mathrm{i}\,\delta(X) U_1(x_1) \tag{3.85}$$

we do not affect the symmetry properties and the field structure (3.24) and we do not spoil the previous three finite renormalizations. As a result we get

$$P_1(X) = 0 \tag{3.86}$$

and the proof is complete. □

Remark 3.2. One can show that one can give up the induction hypothesis (3.16). Indeed, some new terms (containing a factor $\partial_\rho A_\sigma$) appear in the anomalies and they can be eliminated using the same properties as before.

Remark 3.3. It is easy to see that the same pattern works for scalar electrodynamics also. A minor modification appears for the expression of W_1: the terms $c_4 - c_7$ must be replaced by

$$W_1 = c_4 : u\bar{\phi}\phi : + c_5 : uA_\mu\bar{\phi}\partial^\mu\phi : + c_6 : uA_\mu\partial^\mu\bar{\phi}\phi : \tag{3.87}$$

The first contribution is cancelled by charge conjugation invariance and the last two by the condition (3.79).

4. The structure of the anomalies in higher orders for the Yang–Mills model

4.1. The anomalous gauge equations

We give now the results for the Yang–Mills model as presented in section 2.2. By comparison to the case of QED, two important modifications appear. The first one is the relation (3.11) which is replaced by

$$d_Q T^\mu(x) = \mathrm{i}\frac{\partial}{\partial x^\nu} T^{\mu\nu}(x) \tag{4.1}$$

where

$$T^{\mu\nu}(x) \equiv \frac{1}{2} f_{abc} : u_a u_b F_c^{\mu\nu} : \tag{4.2}$$

Let us note the antisymmetry property

$$T^{\mu\nu}(x) = -T^{\nu\mu}(x) \tag{4.3}$$

and the analogue of (3.11)

$$d_Q T^{\mu\nu}(x) = 0. \tag{4.4}$$

We also have

$$gh(T^{\mu\nu}(x)) = 2. \tag{4.5}$$

The second change is the disappearance of charge conjugation invariance. Because of these changes we will not be able to prove the disappearance of the anomalies in higher orders of perturbation theory. Instead, we will be able to give the generic structure of these anomalies. The computations are similar to those from the preceding section but are more complicated from the combinatorial point of view. Because there are no essential new subtleties we will give only the results.

There is an important case when the charge conjugation invariance survives, namely in a pure Yang–Mills theory without γ_5-couplings for the Dirac fields. This means that in the formulæ from section 2.2 one should consider only fields of type (III), i.e. the scalar ghosts are absent (this implies that we should also take the scalar current to be null $j_a = 0, \forall a$ and $m_a = 0, \forall a$) and the structure of the current (2.27) is much simpler: $t'_a = 0, \forall a$, i.e. no axial interaction is allowed. In this case, if the gauge group is $SU(N)$ one can prove that charge conjugation invariance exists [19, 28]. Namely for any simple compact Lie algebra one can find a matrix U_{ab} such that we have

$$U_{ab} t_b = -t_a^T \qquad \forall a; \tag{4.6}$$

then, similarly to (3.2) we define the operator U_C through the relations

$$\begin{aligned} &U_C A_a^\mu(x) U_C^{-1} = U_{ab} A_b^\mu(x) \qquad U_C u_a(x) U_C^{-1} = U_{ab} u_b(x) \\ &U_C \tilde{u}_a(x) U_C^{-1} = U_{ab} \tilde{u}_b(x) \qquad U_C \psi_A(x) U_C^{-1} = \gamma_0\gamma_2 \bar{\psi}_A(x)^T \qquad U_C \Omega = \Omega \end{aligned} \tag{4.7}$$

and one can easily check that we still have relations of the type (3.10) for the expressions $T(x), T^\mu(x), T^{\mu\nu}$ restricted by the previous conditions (pure Yang–Mills and no γ_5 interaction):

$$U_C T(x) U_C^{-1} = T(x) \qquad U_C T^\mu(x) U_C^{-1} = T^\mu(x) \qquad U_C T^{\mu\nu}(x) U_C^{-1} = T^{\mu\nu}(x). \tag{4.8}$$

Then one can prove, exactly as for QED that in this particular case we have the charge conjugation invariance of the anomalies (3.25):

$$U_C P^{k_1,\ldots,k_n}(X) U_C^{-1} = P^{k_1,\ldots,k_n}(X). \tag{4.9}$$

We will prove later that in this case the anomalies can be eliminated.

Like in the case of QED we are looking for the Wess–Zumino consistency relations; for this purpose we denote by $A^k(x),\ k = 1, \ldots, 11$ the expressions $T(x),\ T^\mu(x),\ T^{\mu\nu}$; the index k can take the values $L, \mu, \mu\nu$ according to the identifications $A^L(x) \equiv T(x)$, $A^\mu(x) \equiv T^\mu(x)$, $A^{\mu\nu}(x) \equiv T^{\mu\nu}(x)$. Then we can write, in analogy to (3.12):

$$d_Q A^k(x) = \mathrm{i} \sum_m c_m^{k;\mu} \frac{\partial}{\partial x^\mu} A^m(x) \qquad k = 1, \ldots, 11 \tag{4.10}$$

for some constants $c_m^{k;\mu}$; the explicit expressions are

$$c_\nu^{L;\mu} \equiv \delta_\nu^\mu \qquad c_{\rho\sigma}^{\nu;\mu} \equiv \frac{1}{2}\left(\delta_\rho^\nu \delta_\sigma^\mu - \delta_\rho^\mu \delta_\sigma^\nu\right) \tag{4.11}$$

and the others are zero. Then we conjecture the following result: one can chose the chronological products such that, beside the fulfilment of the Bogoliubov axioms, the following identities are verified:

$$d_Q T(A^{k_1}(x_1), \ldots, A^{k_p}(x_p)) = \mathrm{i} \sum_{l=1}^{p} (-1)^{s_l} \times \sum_m c_m^{k_l;\mu} \frac{\partial}{\partial x_l^\mu} T(A^{k_1}(x_1), \ldots, A^m(x_l), \ldots, A^{k_p}(x_p)) \tag{4.12}$$

for all $p \in \mathbb{N}$ and all $k_1, \ldots, k_p = 1, \ldots, 11$. Here the expression s_l has the same significance as in the case of QED.

There are a number of facts which can be proved identically. First one can prove by induction that one can choose the chronological products such that one has (3.17), the symmetry property (3.18) and

$$T(T^{\mu\nu}(x_1), A_2(x_2), \ldots, A_n(x_n)) = -T(T^{\nu\mu}(x_1), A_2(x_2), \ldots, A_n(x_n)). \tag{4.13}$$

Next, we can prove that the chronological product can be chosen to depend on the following fields:

$$A_a^\mu,\ F_a^{\mu\nu},\ u_a,\ \tilde{u}_a,\ \partial_\mu \tilde{u}_a,\ \Phi_a,\ \partial_\mu \Phi_a,\ \psi_A,\ \overline{\psi}_A. \tag{4.14}$$

Suppose that we have proved the identity (4.12) up to the order $n-1$; then in order n we must have a relation of the type (3.19) where $P_{\ldots}(X) \equiv P_{\ldots}(x_1, \ldots, x_n)$ are the anomalies having the structure (3.20). The maximal degree of the anomaly is also restricted by (3.22) and we still have the constraint (3.23) coming from the ghost number counting. The anomalies will depend on the following set of fields:

$$A_a^\mu,\ \partial_\mu A_a^\nu,\ \partial_\rho F_a^{\mu\nu},\ u_a,\ \partial_\mu u_a,\ \tilde{u}_a,\ \partial_\mu \tilde{u}_a,\ \partial_\mu \partial_\nu \tilde{u}_a,\ \Phi_a,\ \partial_\mu \Phi_a,\ \partial_\mu \partial_\nu \Phi_a,\ \psi_A,\ \partial_\mu \psi_A,\ \overline{\psi}_A,\ \partial_\mu \overline{\psi}_a \tag{4.15}$$

and the factor $\partial_\mu u_a$ can appear only once in any Wick term of the anomaly. Finally, the anomalies can be chosen $SL(2, \mathbb{C})$-covariant.

From the restrictions (3.22) and (3.23) we obtain that the possible anomalies can appear in the following relations:

$$d_Q T(T(x_1), \ldots, T(x_n)) = \mathrm{i} \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} T(T(x_1), \ldots, T^\mu(x_l), \ldots, T(x_n)) + P_1(x_1, \ldots, x_n) \tag{4.16}$$

$$d_Q T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) = \mathrm{i} \frac{\partial}{\partial x_1^\mu} T(T^{\mu\nu}(x_1), T(x_2), \ldots, T(x_n)) - \mathrm{i} \sum_{l=2}^{n} \frac{\partial}{\partial x_l^\nu} T(T^\mu(x_1), T(x_2), \ldots, T^\nu(x_l), \ldots, T(x_n)) + P_2^\mu(x_1, \ldots, x_n) \tag{4.17}$$

$$\begin{aligned} d_Q T(T^\mu(x_1), T^\nu(x_2), T(x_3), \ldots, T(x_n)) &= \mathrm{i} \frac{\partial}{\partial x_1^\rho} T(T^{\mu\rho}(x_1), T^\nu(x_2), T(x_3), \ldots, T(x_n)) \\ &\quad - \mathrm{i} \frac{\partial}{\partial x_2^\rho} T(T^\mu(x_1), T^{\nu\rho}(x_2), T(x_3), \ldots, T(x_n)) \\ &\quad + \mathrm{i} \sum_{l=3}^{n} \frac{\partial}{\partial x_l^\rho} T(T^\mu(x_1), T^\nu(x_2), T(x_3), \ldots, T^\rho(x_l), \ldots, T(x_n)) \\ &\quad + P_3^{\mu\nu}(x_1, \ldots, x_n) \end{aligned} \tag{4.18}$$

$$
\begin{aligned}
& d_Q T(T^{\mu\nu}(x_1), T(x_2), \ldots, T(x_n)) \\
&\qquad = \mathrm{i} \sum_{l=2}^{n} \frac{\partial}{\partial x_l^{\rho}} T(T^{\mu\nu}(x_1), T(x_2), \ldots, T^{\rho}(x_l), \ldots, T(x_n)) \\
&\qquad\quad + P_4^{\mu\nu}(x_1, \ldots, x_n)
\end{aligned} \tag{4.19}
$$

$$
\begin{aligned}
& d_Q T(T^{\mu\nu}(x_1), T^{\rho}(x_2), T(x_3), \ldots, T(x_n)) \\
&\qquad = \mathrm{i} \frac{\partial}{\partial x_2^{\sigma}} T(T^{\mu\nu}(x_1), T^{\rho\sigma}(x_2), T(x_3), \ldots, T(x_n)) \\
&\qquad\quad - \mathrm{i} \sum_{l=3}^{n} \frac{\partial}{\partial x_l^{\sigma}} T(T^{\mu\nu}(x_1), T^{\rho}(x_2), \ldots, T^{\sigma}(x_l), \ldots, T(x_n)) \\
&\qquad\quad + P_5^{\mu\nu\rho}(x_1, \ldots, x_n)
\end{aligned} \tag{4.20}
$$

$$
\begin{aligned}
& d_Q T(T^{\mu}(x_1), T^{\nu}(x_2), T^{\rho}(x_3), T(x_4), \ldots, T(x_n)) \\
&\qquad = \mathrm{i} \frac{\partial}{\partial x_1^{\sigma}} T(T^{\mu\sigma}(x_1), T^{\nu}(x_2), T^{\rho}(x_3), T(x_4), \ldots, T(x_n)) \\
&\qquad\quad - \mathrm{i} \frac{\partial}{\partial x_2^{\sigma}} T(T^{\mu}(x_1), T^{\nu\sigma}(x_2), T^{\rho}(x_3), T(x_4), \ldots, T(x_n)) \\
&\qquad\quad + \mathrm{i} \frac{\partial}{\partial x_3^{\sigma}} T(T^{\mu}(x_1), T^{\nu}(x_2), T^{\rho\sigma}(x_3), T(x_4), \ldots, T(x_n)) \\
&\qquad\quad - \mathrm{i} \sum_{l=4}^{n} \frac{\partial}{\partial x_l^{\sigma}} T(T^{\mu}(x_1), T^{\nu}(x_2), T^{\rho}(x_3), T(x_4), \ldots, T^{\sigma}(x_l), \ldots, T(x_n)) \\
&\qquad\quad + P_6^{\mu\nu\rho}(x_1, \ldots, x_n)
\end{aligned} \tag{4.21}
$$

$$
\begin{aligned}
& d_Q T(T^{\mu\nu}(x_1), T^{\rho\sigma}(x_2), T(x_3), \ldots, T(x_n)) \\
&\qquad = \mathrm{i} \sum_{l=3}^{n} \frac{\partial}{\partial x_l^{\lambda}} T(T^{\mu\nu}(x_1), T^{\rho\sigma}(x_2), T(x_3), \ldots, T^{\lambda}(x_l), \ldots, T(x_n)) \\
&\qquad\quad + P_7^{\mu\nu\rho\sigma}(x_1, \ldots, x_n)
\end{aligned} \tag{4.22}
$$

$$
\begin{aligned}
& d_Q T(T^{\mu\nu}(x_1), T^{\rho}(x_2), T^{\sigma}(x_3), T(x_4), \ldots, T(x_n)) \\
&\qquad = \mathrm{i} \frac{\partial}{\partial x_2^{\lambda}} T(T^{\mu\nu}(x_1), T^{\rho\lambda}(x_2), T^{\sigma}(x_3), T(x_4), \ldots, T(x_n)) \\
&\qquad\quad - \mathrm{i} \frac{\partial}{\partial x_3^{\lambda}} T(T^{\mu\nu}(x_1), T^{\rho}(x_2), T^{\sigma\lambda}(x_3), T(x_4), \ldots, T(x_n)) \\
&\qquad\quad + \mathrm{i} \sum_{l=4}^{n} \frac{\partial}{\partial x_l^{\lambda}} T(T^{\mu\nu}(x_1), T^{\rho}(x_2), T^{\sigma}(x_3), T(x_4), \ldots, T^{\lambda}(x_l), \ldots, T(x_n)) \\
&\qquad\quad + P_8^{\mu\nu\rho\sigma}(x_1, \ldots, x_n)
\end{aligned} \tag{4.23}
$$

$$
\begin{aligned}
& d_Q T(T^{\mu}(x_1), T^{\nu}(x_2), T^{\rho}(x_3), T^{\sigma}(x_4), \ldots, T(x_n)) \\
&\qquad = \mathrm{i} \frac{\partial}{\partial x_1^{\lambda}} T(T^{\mu\lambda}(x_1), T^{\nu}(x_2), T^{\rho}(x_3), T^{\sigma}(x_4), T(x_5), \ldots, T(x_n)) \\
&\qquad\quad - \mathrm{i} \frac{\partial}{\partial x_2^{\lambda}} T(T^{\mu}(x_1), T^{\nu\lambda}(x_2), T^{\rho}(x_3), T^{\sigma}(x_4), T(x_5), \ldots, T(x_n)) \\
&\qquad\quad + \mathrm{i} \frac{\partial}{\partial x_3^{\lambda}} T(T^{\mu}(x_1), T^{\nu}(x_2), T^{\rho\lambda}(x_3), T^{\sigma}(x_4), T(x_5), \ldots, T(x_n))
\end{aligned}
$$

$$- \mathrm{i}\frac{\partial}{\partial x_4^\lambda} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^{\sigma\lambda}(x_4), T(x_5), \ldots, T(x_n))$$
$$+ \mathrm{i}\sum_{l=5}^n \frac{\partial}{\partial x_l^\lambda} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), T(x_5), \ldots, T^\lambda(x_l), \ldots, T(x_n))$$
$$+ P_9^{\mu\nu\rho\lambda}(x_1, \ldots, x_n) \tag{4.24}$$

where we can assume that

$$\begin{array}{llll} P_3^{\mu\nu}(X) = 0 & P_5^{\mu\nu\rho} = 0 & P_7^{\mu\nu\rho\sigma} = 0 & |X| = 1 \\ P_6^{\mu\nu\rho}(X) = 0 & P_8^{\mu\nu\rho\sigma} = 0 & |X| \leqslant 2 & \\ P_9^{\mu\nu\rho\sigma}(X) = 0 & |X| \leqslant 3 & & \end{array} \tag{4.25}$$

without losing generality.

From (3.18), we get the following symmetry properties:

$$P_1(x_1, \ldots, x_n) \text{ is symmetric in } x_1, \ldots, x_n \tag{4.26}$$
$$P_2^\mu(x_1, \ldots, x_n) \text{ is symmetric in } x_2, \ldots, x_n \tag{4.27}$$
$$P_3^{\mu\nu}(x_1, \ldots, x_n) \text{ is symmetric in } x_3, \ldots, x_n \tag{4.28}$$
$$P_4^{\mu\nu}(x_1, \ldots, x_n) \text{ is symmetric in } x_2, \ldots, x_n \tag{4.29}$$
$$P_5^{\mu\nu\rho}(x_1, \ldots, x_n) \text{ is symmetric in } x_3, \ldots, x_n \tag{4.30}$$
$$P_6^{\mu\nu\rho}(x_1, \ldots, x_n) \text{ is symmetric in } x_4, \ldots, x_n \tag{4.31}$$
$$P_7^{\mu\nu\rho\sigma}(x_1, \ldots, x_n) \text{ is symmetric in } x_3, \ldots, x_n \tag{4.32}$$
$$P_8^{\mu\nu\rho\sigma}(x_1, \ldots, x_n) \text{ is symmetric in } x_4, \ldots, x_n \tag{4.33}$$
$$P_9^{\mu\nu\rho\sigma}(x_1, \ldots, x_n) \text{ is symmetric in } x_5, \ldots, x_n \tag{4.34}$$

we also have

$$P_3^{\mu\nu}(x_1, \ldots, x_n) \text{ is antisymmetric in } (x_1, \mu), (x_2, \nu) \tag{4.35}$$
$$P_4^{\mu\nu} = -P_4^{\nu\mu} \tag{4.36}$$
$$P_5^{\mu\nu\rho} = -P_5^{\nu\mu\rho} \tag{4.37}$$
$$P_6^{\mu\nu\rho}(x_1, \ldots, x_n) \text{ is antisymmetric in } (x_1, \mu), (x_2, \nu), (x_3, \rho) \tag{4.38}$$
$$P_7^{\mu\nu\rho\sigma} = -P_7^{\nu\mu\rho\sigma} = -P_7^{\mu\nu\sigma\rho} \tag{4.39}$$
$$P_7^{\mu\nu\rho\sigma}(x_1, x_2, \ldots, x_n) = P_7^{\rho\sigma\mu\nu}(x_2, x_1, \ldots, x_n) \tag{4.40}$$
$$P_8^{\mu\nu\rho\sigma} = -P_8^{\nu\mu\rho\sigma} \tag{4.41}$$
$$P_8^{\mu\nu\rho\sigma}(x_1, x_2, x_3, \ldots, x_n) = -P_8^{\mu\nu\sigma\rho}(x_1, x_3, x_2, \ldots, x_n) \tag{4.42}$$
$$P_9^{\mu\nu\rho\sigma}(x_1, \ldots, x_n) \quad \text{is antisymmetric in } (x_1, \mu), (x_2, \nu), (x_3, \rho), (x_4, \sigma). \tag{4.43}$$

Let us note that for $n = 2$ only the first five relations (4.16)–(4.20) have to be checked; this can be done by some long but straightforward computations.

4.2. The generic structure of the anomalies

If we apply the operator d_Q to the anomalous relations (4.16)–(4.24) we easily obtain again consistency relations of the Wess–Zumino type:

$$d_Q P_1(x_1, \ldots, x_n) = -\mathrm{i}\sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} P_2^\mu(x_l, x_1, \ldots, \hat{x}_l, \ldots, x_n) \tag{4.44}$$

$$d_Q P_2^{\mu}(x_1,\ldots,x_n) = -\mathrm{i}\frac{\partial}{\partial x_1^{\nu}} P_4^{\mu\nu}(x_1,\ldots,x_n) + \mathrm{i}\sum_{l=2}^{n}\frac{\partial}{\partial x_l^{\nu}} P_3^{\mu\nu}(x_1,x_l,x_2,\ldots,\hat{x}_l,\ldots,x_n) \tag{4.45}$$

$$d_Q P_3^{\mu\nu}(x_1,\ldots,x_n) = -\mathrm{i}\frac{\partial}{\partial x_1^{\rho}} P_5^{\mu\rho\nu}(x_1,\ldots,x_n) + \mathrm{i}\frac{\partial}{\partial x_2^{\rho}} P_5^{\nu\rho\mu}(x_2,x_1,x_3,\ldots,x_n) - \mathrm{i}\sum_{l=3}^{n}\frac{\partial}{\partial x_l^{\rho}} P_4^{\mu\nu\rho}(x_1,x_2,x_l,x_3,\ldots,\hat{x}_l,\ldots,x_n) \tag{4.46}$$

$$d_Q P_4^{\mu\nu}(x_1,\ldots,x_n) = -\mathrm{i}\sum_{l=2}^{n}\frac{\partial}{\partial x_l^{\rho}} P_5^{\mu\nu\rho}(x_1,x_l,x_2,\ldots,\hat{x}_l,\ldots,x_n) \tag{4.47}$$

$$d_Q P_5^{\mu\nu\rho}(x_1,\ldots,x_n) = -\mathrm{i}\frac{\partial}{\partial x_2^{\sigma}} P_7^{\mu\nu\rho\sigma}(x_1,\ldots,x_n) + \mathrm{i}\sum_{l=3}^{n}\frac{\partial}{\partial x_l^{\sigma}} P_8^{\mu\nu\rho\sigma}(x_1,x_2,x_l,x_3,\ldots,\hat{x}_l,\ldots,x_n) \tag{4.48}$$

$$d_Q P_6^{\mu\nu\rho}(x_1,\ldots,x_n) = -\mathrm{i}\frac{\partial}{\partial x_1^{\sigma}} P_8^{\mu\sigma\nu\rho}(x_1,\ldots,x_n) + \mathrm{i}\frac{\partial}{\partial x_2^{\sigma}} P_8^{\nu\sigma\mu\rho}(x_2,x_1,x_3,\ldots,x_n) - \mathrm{i}\frac{\partial}{\partial x_3^{\sigma}} P_8^{\rho\sigma\mu\nu}(x_3,x_2,x_1,x_4,\ldots,x_n) + \mathrm{i}\sum_{l=4}^{n}\frac{\partial}{\partial x_l^{\rho}} P_9^{\mu\nu\rho\sigma}(x_1,x_2,x_3,x_l,x_4,\ldots,\hat{x}_l,\ldots,x_n) \tag{4.49}$$

$$d_Q P_i^{\mu\nu\rho\sigma}(x_1,\ldots,x_n) = 0 \qquad i = 7,8,9. \tag{4.50}$$

After a long computation (using the symmetry properties, the ghost number restrictions, etc and making some convenient finite renormalizations) one can determine the generic form of the anomalies. One starts from a generic form of the same type as in the case of QED for all anomalies $P_i^{\cdots}$, $i = 1,\ldots,9$ and determines that

$$P_i^{\cdots} = 0 \qquad i = 3,\ldots,9 \tag{4.51}$$

and $P_i^{\cdots}$, $i = 1,2$ can be chosen of the form

$$P_1 = \delta(X)W_1(x_1) \qquad P_2^{\mu} = \delta(X)W_2^{\mu}(x_1). \tag{4.52}$$

The gauge invariance condition reduces to

$$d_Q W_1 = \mathrm{i}\partial_\mu W_2^{\mu} \qquad d_Q W_2^{\mu} = 0. \tag{4.53}$$

We now give the generic form of the Wick polynomials W_i, $i = 1,2$ fulfilling these conditions. First we have

$$W_2^{\mu} = c_{abcd} :u_a u_b \Phi_c \partial^{\mu}\Phi_d: + c_{abc} :u_a u_b \partial^{\mu}\Phi_c: + c_{ab;AB} :u_a u_b \overline{\psi}_A \gamma^{\mu}\psi_B: + c'_{ab;AB} :u_a u_b \overline{\psi}_A \gamma^{\mu}\gamma_5\psi_B: \tag{4.54}$$

where

$$\begin{aligned} &c_{abcd} = -(c\leftrightarrow d) = -(a\leftrightarrow b) \qquad c_{ab;AB} = -(a\leftrightarrow b) \qquad c'_{ab;AB} = -(a\leftrightarrow b) \\ &c_{abcd} = 0 \quad m_c + m_d \geqslant 0 \qquad c_{abc} = 0 \quad m_a + m_b + m_c \geqslant 0. \end{aligned} \tag{4.55}$$

Finally we have

$$
\begin{aligned}
W_1 = 2c_{abcd} &: u_a A^\rho_b \Phi_c \partial_\rho \Phi_d : + 2c_{abc} : u_a A^\rho_b \partial_\rho \Phi_c : + 2c_{ab;AB} : u_a A^\rho_b \overline{\psi}_A \gamma_\rho \psi_B : \\
&+ 2c'_{ab;AB} : u_a A^\rho_b \overline{\psi}_A \gamma_\rho \gamma_5 \psi_B : + \mathrm{i} \sum_{m_b \neq 0} \frac{1}{m_b} (M_A - M_B) c_{ab;AB} : u_a \Phi_b \overline{\psi}_A \psi_B : \\
&+ \mathrm{i} \sum_{m_b \neq 0} \frac{1}{m_b} (M_A + M_B) c'_{ab;AB} : u_a \Phi_b \overline{\psi}_A \gamma_5 \psi_B : \\
&+ g'_{abc} \left(: u_a \partial_\rho \Phi_b \partial^\rho \Phi_c : + m_b m_c : u_a A^\rho_b A_{c\rho} : - 2m_b : u_a A^\rho_b \partial_\rho \Phi_c : \right) \\
&+ g_{abc} : u_a \Phi_b \Phi_c : + g_{abcd} : u_a \Phi_b \Phi_c \Phi_d : + g_{abcde} : u_a \Phi_b \Phi_c \Phi_d \Phi_e : \\
&+ d_{a;AB} : u_a \overline{\psi}_A \psi_B : + d'_{a;AB} : u_a \overline{\psi}_A \gamma_5 \psi_B : \\
&+ h_{abc} : u_a F^{\rho\sigma}_b F_{c\rho\sigma} : + h'_{abc} \varepsilon_{\mu\nu\rho\sigma} : u_a F^{\mu\nu}_b F^{\rho\sigma}_c :
\end{aligned}
\tag{4.56}
$$

where

$$
\begin{aligned}
&g_{abc} = 0 \quad m_a + m_b + m_c > 0 \qquad g_{abcd} = 0 \quad m_a + m_b + m_c + m_d > 0 \\
&h_{abc} = 0 \quad h'_{abc} = 0 \quad m_a > 0 \qquad d_{a;AB} = 0 \quad d'_{a;AB} = 0 \quad m_a > 0.
\end{aligned}
\tag{4.57}
$$

One can prove that these anomalies cannot be eliminated by further redefinitions of the chronological products (so they are representatives modulo anomalies of the type $d_Q A + \partial_\mu A^\mu$); it follows there are no obvious arguments for the elimination of these anomalies. We remark upon a very interesting fact: if all the bosons are heavy, then there the expression of the anomalies simplifies considerably. Also in the case when only massless bosons are present the expression of the anomaly simplifies drastically. In fact, in this case, only indices of type (IIb) are present (see section 2.2), i.e. we do not have the scalar fields Φ_a at all. Moreover, all masses m_a are null. One can obtain quite easily the structure of the anomalies in this case:

$$
W^\mu_2 = c_{ab;AB} : u_a u_b \overline{\psi}_A \gamma^\mu \psi_B : + c'_{ab;AB} : u_a u_b \overline{\psi}_A \gamma^\mu \gamma_5 \psi_B :
\tag{4.58}
$$

and

$$
\begin{aligned}
W_1 = 2c_{ab;AB} &: u_a A^\rho_b \overline{\psi}_A \gamma_\rho \psi_B : + 2c'_{ab;AB} : u_a A^\rho_b \overline{\psi}_A \gamma_\rho \gamma_5 \psi_B : + d_{a;AB} : u_a \overline{\psi}_A \psi_B : \\
&+ d'_{a;AB} : u_a \overline{\psi}_A \gamma_5 \psi_B : + h_{abc} : u_a F^{\rho\sigma}_b F_{c\rho\sigma} : + h'_{abc} \varepsilon_{\mu\nu\rho\sigma} : u_a F^{\mu\nu}_b F^{\rho\sigma}_c :
\end{aligned}
\tag{4.59}
$$

with the following restrictions:

$$
\begin{aligned}
&c_{ab;AB} = -(a \leftrightarrow b) \qquad c'_{ab;AB} = -(a \leftrightarrow b) \\
&(M_A - M_B) c_{ab;AB} = 0 \qquad (M_A + M_B) c'_{ab;AB} = 0.
\end{aligned}
\tag{4.60}
$$

We have essentially obtained the results presented in [27,28] and [17] in a purely algebraic way. In [27, 28] it is argued that in some cases the uFF anomalies are absent and in [17] the argument is extended to the Dirac contributions. This can be done if we assume that the gauge group is $SU(N)$ and no axial interaction in allowed. Let us give the proof of the same result using our formalism. First, one can prove by induction that the γ_5 terms from the preceding expressions are also absent, i.e. we have

$$
W^\mu_2 = c_{ab;AB} : u_a u_b \overline{\psi}_A \gamma^\mu \psi_B :
\tag{4.61}
$$

and

$$
\begin{aligned}
W_1 = 2c_{ab;AB} &: u_a A^\rho_b \overline{\psi}_A \gamma_\rho \psi_B : + d_{a;AB} : u_a \overline{\psi}_A \psi_B : \\
&+ h_{abc} : u_a F^{\rho\sigma}_b F_{c\rho\sigma} : + h'_{abc} \varepsilon_{\mu\nu\rho\sigma} : u_a F^{\mu\nu}_b F^{\rho\sigma}_c :
\end{aligned}
\tag{4.62}
$$

Now we can also use an argument based on charge conjugation invariance. We refer to some group-theoretical results proved in [28] and [17]. First, one can prove that the expressions h_{abc} and h'_{abc}, respectively, must be linear combinations of the completely antisymmetric and the completely symmetric tensors f_{abc} and d_{abc}. Because of the obvious symmetry property of the constants h_{abc} and h'_{abc} they are necessarily proportional to d_{abc}. But in this case one uses the well known identity

$$U_{aa'}U_{bb'}U_{cc'}d_{a'b'c'} = -d_{abc} \tag{4.63}$$

and shows that the charge conjugation invariance (4.8) implies that we must have $h_{abc} = h'_{abc} = 0$; (this is a particular case of the even–odd Furry theorem from the appendix of [28]). Other group-theoretical considerations give the structure of the $N \times N$-matrices with elements $d_{a;AB}$ and $c_{ab;AB}$, namely we must have $d_a = Ct_a, c_{ab} = C' f_{abc}t_c$ (see (A.11) and (A.12) of [17]); in the last case we have used the antisymmetry property in the two indices. Now if we use the relation (4.6) and

$$U_{aa'}U_{bb'}U_{cc'}f_{a'b'c'} = f_{abc} \tag{4.64}$$

we find out that we also have $d_a = 0, c_{ab} = 0$.

The result of this discussion reproduces the very important results of [28] and [17]: the pure Yang–Mills model with $SU(N)$ gauge group and without axial interaction is gauge-invariant in all orders of perturbation theory, i.e. one can chose the chronological products such that the relations (4.12) are fulfilled for any $n \in \mathbb{N}^*$. For an arbitrary model with spontaneously broken symmetry and with axial coupling the previous result cannot be obtained by purely algebraic considerations based on Wess–Zumino consistency relations.

We close with another interesting remark connected to the geometric interpretation of the anomalies. Let us define the following differential forms:

$$\mathcal{T}_p(X) \equiv \sum T(A^{k_1}(x_1), \ldots, A^{k_p}(x_p))\, \mathrm{d}x_{1;k_1} \wedge \cdots \wedge \mathrm{d}x_{p;k_p} \tag{4.65}$$

where, in general, we have defined

$$\mathrm{d}x_L \equiv \mathrm{d}x \equiv \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3 \qquad \mathrm{d}x_\mu \equiv \mathrm{i}_{\partial^\mu}\, \mathrm{d}x \qquad \mathrm{d}x_{\rho\sigma} \equiv \mathrm{i}_{\partial^\rho}\mathrm{i}_{\partial^\sigma}\, \mathrm{d}x. \tag{4.66}$$

It is a very interesting fact that the following relation is true:

$$\mathrm{d}x^\rho \wedge \mathrm{d}x_i = \sum_j c_i^{j;\rho}\mathrm{d}x_j \tag{4.67}$$

where the constants $c_i^{j;\rho}$ are exactly the same as those appearing in (4.11). Then it is easy to prove that the induction hypothesis can be compactly written as

$$\mathrm{d}_Q\mathcal{T}_p(X) = \mathrm{i}\, \mathrm{d}\mathcal{T}_p(X) \qquad p = 1, \ldots, n-1 \tag{4.68}$$

and the anomalous gauge identity in order n is

$$\mathrm{d}_Q\mathcal{T}_n(X) = \mathrm{i}\, \mathrm{d}\mathcal{T}_n(X) + \mathcal{P}_n(X) \tag{4.69}$$

here the anomaly $\mathcal{P}_n(X)$ has an expression of the type (4.65)

$$\mathcal{P}_p(X) \equiv \sum P^{k_1,\ldots,k_p}(X)\, \mathrm{d}x_{1;k_1} \wedge \cdots \wedge \mathrm{d}x_{p;k_p} \tag{4.70}$$

with the identifications

$$\begin{array}{lll} P^{L,\ldots,L} = P_1 & P^{\mu,L,\ldots,L} = P_2^\mu & P^{\mu,\nu,L,\ldots,L} = P_3^{\mu\nu} \\ P^{\mu\nu,L,\ldots,L} = P_4^{\mu\nu} & P^{\mu\nu,\rho,L,\ldots,L} = P_5^{\mu\nu\rho} & P^{\mu,\nu,\rho,L,\ldots,L} = P_6^{\mu\nu\rho} \\ P^{\mu\nu,\rho\sigma,L,\ldots,L} = P_7^{\mu\nu\rho\sigma} & P^{\mu\nu,\rho,\sigma,L,\ldots,L} = P_8^{\mu\nu\rho\sigma} & P^{\mu,\nu,\rho,\sigma,L,\ldots,L} = P_9^{\mu\nu\rho\sigma}. \end{array} \tag{4.71}$$

So, the expressions $\mathcal{P}_p(X)$ are differential forms with coefficients quasi-local operators. Let us denote by $\mathcal{A}$ this class of differential forms. From (4.69) we easily obtain the consistency equation

$$\mathrm{d}_Q\mathcal{P}_n(X) + \mathrm{i}\, \mathrm{d}\mathcal{P}_n(X) = 0 \tag{4.72}$$

which is the compact form of the relations (4.44)–(4.50). A more explicit form is

$$\mathrm{d}_Q P^{i_1,\ldots,i_n}(X) = -\mathrm{i}\sum_{l=1}^{n}(-1)^{s_l} c_k^{i_l;\mu} \frac{\partial}{\partial x_l^\mu} P^{i_1,\ldots,i_{l-1},k,i_{l+1},\ldots,i_n}(X). \tag{4.73}$$

One can 'solve' this equation using the homotopy operator p of the de Rham complex: we have

$$\mathcal{P}_n(X) = \mathrm{d}(p\mathcal{P}_n(X)) + \mathrm{i}\,\mathrm{d}_Q(p\mathcal{P}_n(X)). \tag{4.74}$$

It is tempting to argue that by the finite renormalization

$$\mathcal{T}_n(X) \to \mathcal{T}_n(X) + \mathrm{i}p\mathcal{P}_n(X) \tag{4.75}$$

the anomalies are eliminated. However, one can check that if we apply the homotopy operator p on an element from $\mathcal{A}$ we do not obtain a element from $\mathcal{A}$. It follows that the finite renormalization given above is not legitimate and the argument has to be modified somehow. However, let us notice the interesting fact that the usual expression of the homotopy operator for the de Rham complex is constructed using the action of the dilation group. This is in agreement to the role played by this group in the traditional approach to the non-renormalizability theorems.

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